Practical and theoretical aspects of high-dimensional optimal portfolio selection

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Introduction

Optimization problem:

$$\mathbf{w}'\boldsymbol{\mu} - \frac{\gamma}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \rightarrow max \quad \text{subject to} \quad \mathbf{w}'\mathbf{1} = 1\,.$$

where $\boldsymbol{\mu} = \mathrm{E}(\mathbf{y})$, $\boldsymbol{\Sigma} = \mathrm{Var}(\mathbf{y})$, $\mathbf{w} = (w_1, w_2, ..., w_p)'$ is the p-dimensional vector of the portfolio weights, and $\mathbf{y}$ is the vector of asset returns.

Solution:

$$\mathbf{w}_{EU} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} + \gamma^{-1}\mathbf{Q}\boldsymbol{\mu} \ \text{ with } \ \mathbf{Q} = \boldsymbol{\Sigma}^{-1} - \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}\mathbf{1}'\boldsymbol{\Sigma}^{-1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}$$

- Population efficient frontier:

$$(R - R_{GMV})^2 = s(V - V_{GMV})\,,$$

where

$$R_{GMV} = \frac{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}\,, V_{GMV} = \frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}\,, \text{ and } s = \boldsymbol{\mu}'\mathbf{Q}\boldsymbol{\mu}\,.$$

Remark: $\mathbf{w}_{EU}$, R_{EU}, V_{EU}, R_{GMV}, V_{GMV} and s are functions of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ only.

Parameter uncertainty: Estimation

$\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are unknown parameters $\rightsquigarrow$ estimation

Sample mean vector and covariance matrix:

$$\bar{\mathbf{y}}_n = \frac{1}{n}\sum_{j=1}^{n}\mathbf{y}_j \quad \text{and} \quad \mathbf{S}_n = \frac{1}{n-1}\sum_{j=1}^{n}(\mathbf{y}_j - \bar{\mathbf{y}}_n)(\mathbf{y}_j - \bar{\mathbf{y}}_n)'.$$

Sample EU portfolio weights:

$$\hat{\mathbf{w}}_{EU} = \frac{\mathbf{S}_n^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1}} + \alpha^{-1}\hat{\mathbf{Q}}_n\bar{\mathbf{y}}_n \ \text{ with } \ \hat{\mathbf{Q}}_n = \mathbf{S}_n^{-1} - \frac{\mathbf{S}_n^{-1}\mathbf{1}\mathbf{1}'\mathbf{S}_n^{-1}}{\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1}}$$

Sample efficient frontier:

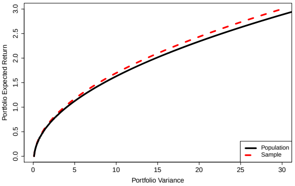
$$(R - \hat{R}_{GMV})^2 = \hat{s}(V - \hat{V}_{GMV}),$$

where

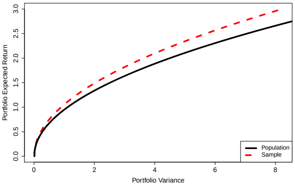
$$\hat{R}_{GMV} = \frac{\mathbf{1}'\mathbf{S}_n^{-1}\bar{\mathbf{y}}_n}{\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1}}, \ \hat{V}_{GMV} = \frac{1}{\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1}}, \ \text{and } \hat{s} = \bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n\bar{\mathbf{y}}_n.$$

How good is the sample efficient frontier?

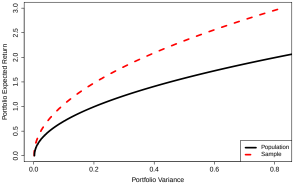
$p = 10$, $n = 500$



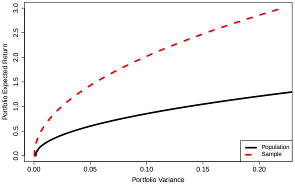
$p = 50$, $n = 500$



$p = 250$, $n = 500$



$p = 400$, $n = 500$



Sample efficient frontier is overoptimistic, especially for large p

Improved estimators: Shrinkage approach

- Mean vector: James and Stein (1961), Gleser (1986), Chételat and Wells (2012), Wang, Tong, Cao and Miao (2014), Bodnar, Okhrin and Parolya (2018)
- Covariance matrix/precision matrix: Efron and Morris (1976), Ledoit and Wolf (2004), Cai, Lui and Luo (2011), Xue and Zou (2012), Fan, Liao and Mincheva (2013), Wang, Pan, Tong and Zhu (2015), Bodnar, Gupta, and Parolya (2014, 2016)
- Optimal portfolio weights: Golosnoy and Okhrin (2007), Frahm and Memmel (2010), Kourtis, Dotsis and Markellos (2012), Bodnar, Parolya, and Schmid (2018)

Classical approach: Optimal portfolio weights are obtained by using improved estimators for the mean vector and for the precision matrix

Here: Improved estimator is constructed directly for portfolio weights $\rightsquigarrow$ Reduction of the estimation error

Data-generating model

Let $\mathbf{Y}_n = (\mathbf{y}_1, ..., \mathbf{y}_n)$ be the observation matrix and let

$$\mathbf{Y}_n \overset{d}{=} \boldsymbol{\mu}\mathbf{1}_n' + \mathbf{\Sigma}^{1/2}\mathbf{X}_n \,,$$

where $\mathbf{X}_n$ consists of i.i.d. random variables with zero means and unit variances.

Assumptions:

(A1) $\mathbf{\Sigma}$ is a non-random positive definite matrix.

(A2) The elements of $\mathbf{X}_n$ have bounded $4 + \varepsilon$ moments for some $\varepsilon > 0$.

(A3) The efficient frontier is asymptotically a nondegenerate object: $s = \boldsymbol{\mu}'\mathbf{Q}\boldsymbol{\mu} > 0$ uniformly in p.

Shrinkage estimator

We seek the estimator in the form (for $c \in (0, +\infty)$, $c \neq 1$)

$$\hat{\mathbf{w}}_{SE} = \alpha_n \hat{\mathbf{w}}_{EU} + (1 - \alpha_n)\mathbf{b} \;\text{ with } \mathbf{b}'\mathbf{1} = 1 \text{ and } \mathbf{b}'\boldsymbol{\Sigma}\mathbf{b} < \infty\,,$$

where

- $\hat{\mathbf{w}}_{EU}$ is the sample estimator of the EU portfolio (Moore-Penrose inverse is used for $c > 1$)
- $\mathbf{b}$ is a vector of given nonrandom target weights with uniformly bounded norm
- α_n – some unknown shrinkage intensity $\rightsquigarrow$ object of interest

Out-of-sample mean-variance objective function:

$$U = \hat{\mathbf{w}}'_{SE}(\alpha_n)\boldsymbol{\mu} - \frac{\gamma}{2}\hat{\mathbf{w}}'_{SE}(\alpha_n)\boldsymbol{\Sigma}\hat{\mathbf{w}}_{SE}(\alpha_n) \longrightarrow max \;\text{ s.t. }\; \alpha_n$$

Solution:

$$\rightsquigarrow \alpha_n^* = \gamma^{-1}\frac{(\hat{\mathbf{w}}_{EU} - \mathbf{b})'(\boldsymbol{\mu} - \gamma\boldsymbol{\Sigma}\mathbf{b})}{(\hat{\mathbf{w}}_{EU} - \mathbf{b})'\boldsymbol{\Sigma}(\hat{\mathbf{w}}_{EU} - \mathbf{b})}\,.$$

Bodnar, T., Okhrin, Y., and Parolya, N. (2023). Optimal shrinkage-based portfolio selection in high dimensions. *Journal of Business & Economic Statistics* **41**: 140-156.

Theorem 1

Under Assumptions (A1)-(A3), it holds that

$$\alpha_n^* \xrightarrow{a.s.} \alpha^* \text{ for } \frac{p}{n} \to c \in (0, +\infty) \setminus \{1\} \text{ as } n \to \infty \text{ with}$$

$$\alpha^* = \begin{cases} \gamma^{-1} \dfrac{(R_{GMV} - R_b)\left(1 + \frac{1}{1-c}\right) + \gamma(V_b - V_{GMV}) + \frac{\gamma^{-1}}{1-c}s}{\frac{1}{1-c}V_{GMV} - 2\left(V_{GMV} + \frac{\gamma^{-1}}{1-c}(R_b - R_{GMV})\right) + \gamma^{-2}\left(\frac{s}{(1-c)^3} + \frac{c}{(1-c)^3}\right) + V_b} & c < 1, \\ \gamma^{-1} \dfrac{(R_{GMV} - R_b)\left(1 + \frac{1}{(c-1)}\right) + \gamma(V_b - V_{GMV}) + \frac{\gamma^{-1}}{c(c-1)}s}{\frac{c^2}{(c-1)}V_{GMV} - 2\left(V_{GMV} + \frac{\gamma^{-1}}{c(c-1)}(R_b - R_{GMV})\right) + \gamma^{-2}\left(\frac{s}{(c-1)^3} + \frac{c^2}{(c-1)^3}\right) + V_b} & c > 1. \end{cases}$$

where

- R_{GMV} and V_{GMV} are the expected return and the variance of the true GMV portfolio
- $R_b = \mathbf{b}'\boldsymbol{\mu}$ and $V_b = \mathbf{b}'\boldsymbol{\Sigma}\mathbf{b}$ are the expected return and the variance of the target portfolio **b**
- s – the slope parameter of the efficient frontier

Oracle shrinkage EU portfolio

$$\hat{\mathbf{w}}_{OSE} = \alpha^* \hat{\mathbf{w}}_{EU} + (1 - \alpha^*)\mathbf{b}$$

Corollary 1

Let U_{SE} and U_S be the expected utilities for the oracle shrinkage EU portfolio and the sample EU portfolio, respectively. Then the relative losses are given by

$$r_{SE} = \frac{U_{EU} - U_{SE}}{U_{EU}} \xrightarrow{a.s.} \begin{cases} (\alpha^*)^2 r_S + (1-\alpha^*)^2 r_{\mathbf{b}} + \alpha^*(1-\alpha^*)\frac{1+c-c^2}{c(c-1)} \frac{R_b - R_{GMV} - \gamma^{-1}s}{U_{EU}} & \text{for } c < 1, \\ (\alpha^*)^2 r_S + (1-\alpha^*)^2 r_{\mathbf{b}} + \alpha^*(1-\alpha^*)\frac{c}{1-c} \frac{R_b - R_{GMV} - \gamma^{-1}s}{U_{EU}} & \text{for } c > 1. \end{cases}$$

$$r_S = \frac{U_{EU} - U_S}{U_{EU}} \xrightarrow{a.s.} \begin{cases} \dfrac{\frac{\gamma}{2}\left(\frac{1}{1-c} - 1\right)\cdot V_{GMV} + \gamma^{-1}\left(\frac{1}{2} - \frac{1}{(1-c)} + \frac{1}{2(1-c)^3}\right)\cdot s + \frac{\gamma^{-1}}{2}\cdot\frac{c}{(1-c)^3}}{R_{GMV} + \frac{\gamma^{-1}}{2}\cdot s - \frac{\gamma}{2}V_{GMV}} & \text{for } c < 1, \\ \dfrac{\frac{\gamma}{2}\left(\frac{c^2}{c-1} - 1\right)\cdot V_{GMV} + \gamma^{-1}\left(\frac{1}{2} - \frac{1}{c(c-1)} + \frac{1}{2(c-1)^3}\right)\cdot s + \frac{\gamma^{-1}}{2}\cdot\frac{c^2}{(1-c)^3}}{R_{GMV} + \frac{\gamma^{-1}}{2}\cdot s - \frac{\gamma}{2}V_{GMV}} & \text{for } c > 1. \end{cases}$$

for $\frac{p}{n} \to c \in (0, +\infty) \setminus \{1\}$ as $n \to \infty$.

Relative losses as a function of p

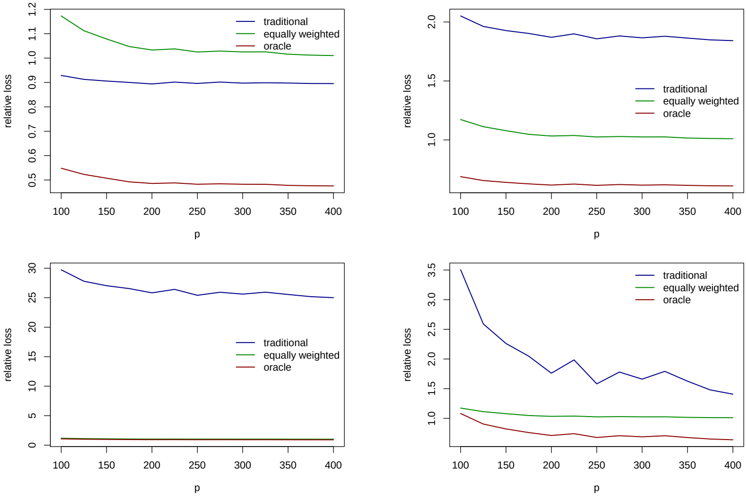


Figure: $c = 0.2$ (top left), 0.5 (top right), 0.8 (bottom left), 2 (bottom right)

Bona-fide estimation

Theorem 2

Under Assumptions (A1)-(A3), consistent estimators for R_{GMV}, V_{GMV}, s, R_b and V_b under large dimensional asymptotics $p/n \to c$ as $n \to \infty$ are given by

$$\begin{aligned}
\hat{R}_c &= \hat{R}_{GMV} \xrightarrow{a.s.} R_{GMV} \\
\hat{V}_c &= \begin{cases} \dfrac{1}{1-p/n}\hat{V}_{GMV} & \text{for } c<1, \\[2ex] \dfrac{1}{p/n(p/n-1)}\hat{V}_{GMV} & \text{for } c>1. \end{cases} \xrightarrow{a.s.} V_{GMV} \\
\hat{s}_c &= \begin{cases} (1-p/n)\hat{s} - p/n & \text{for } c<1, \\[2ex] p/n(p/n-1)\hat{s} - p/n & \text{for } c>1. \end{cases} \xrightarrow{a.s.} s \\
\hat{R}_b &= \mathbf{b}'\bar{\mathbf{y}}_n \xrightarrow{a.s.} R_b \\
\hat{V}_b &= \mathbf{b}'\mathbf{S}_n\mathbf{b} \xrightarrow{a.s.} V_b \,,
\end{aligned}$$

where $\hat{R}_{GMV}$, $\hat{V}_{GMV}$ and $\hat{s}$ are the corresponding sample estimators.

Bona-fide shrinkage EU portfolio

$$\hat{\mathbf{w}}_{BFSE} = \widehat{\alpha}^* \hat{\mathbf{w}}_{EU} + (1 - \widehat{\alpha}^*)\mathbf{b}$$

with

$$\widehat{\alpha}_c^*(\mathbf{b}) = \begin{cases} \gamma^{-1} \dfrac{(\hat{R}_c - \hat{R}_b)\left(1 + \dfrac{1}{1-p/n}\right) + \gamma(\hat{V}_b - \hat{V}_c) + \dfrac{\gamma^{-1}}{1-p/n}\hat{s}_c}{\dfrac{1}{1-p/n}\hat{V}_c - 2\left(\hat{V}_c + \frac{\gamma^{-1}}{1-p/n}(\hat{R}_b - \hat{R}_c)\right) + \gamma^{-2}\left(\dfrac{\hat{s}_c}{(1-p/n)^3} + \dfrac{p/n}{(1-p/n)^3}\right) + \hat{V}_b} & c < 1, \\ \gamma^{-1} \dfrac{(\hat{R}_c - \hat{R}_b)\left(1 + \dfrac{1}{p/n(p/n-1)}\right) + \gamma(\hat{V}_b - \hat{V}_c) + \dfrac{\gamma^{-1}}{p/n(p/n-1)}\hat{s}_c}{\dfrac{(p/n)^2}{p/n-1}\hat{V}_c - 2\left(\hat{V}_c + \frac{\gamma^{-1}}{p/n(p/n-1)}(\hat{R}_b - \hat{R}_c)\right) + \dfrac{\gamma^{-2}}{(p/n-1)^3}\left(\hat{s}_c + (p/n)^2\right) + \hat{V}_b} & c > 1. \end{cases}$$

Tests on the weights of optimal portfolios

Generalized linear hypothesis:

$$H_0 : \mathbf{L}\mathbf{w}_{EU} = \mathbf{r} \qquad \text{against} \qquad H_1 : \mathbf{L}\mathbf{w}_{EU} \neq \mathbf{r},$$

- $\mathbf{L}$: $k \times p$ dimensional matrix with $k < p - 1$
- $\mathbf{r}$: k-dimensional vector

Approach: Tests based on Mahalanobis distance

Bodnar, T. and Schmid, W. (2011). On the exact distribution of the estimated expected utility portfolio weights: Theory and applications, *Statistics & Risk Modeling* **28**: 319-342.

Bodnar, T., Dette, H., Parolya, N. and Thorsén, E. (2022). Sampling distributions of optimal portfolio weights and characteristics in low and large dimensions, *Random Matrices: Theory and Applications* **11**: 2250008.

Test based on Mahalanobis distance

Test statistics:

$$T_{\mathbf{L}} = (n-p+1)\,(\hat{\mathbf{w}}_{\mathbf{L}} - \mathbf{r})'\,\hat{\boldsymbol{\Omega}}_{\mathbf{L}}^{-1}\,(\hat{\mathbf{w}}_{\mathbf{L}} - \mathbf{r})\,,$$

where

$$\hat{\mathbf{w}}_{\mathbf{L}} = \mathbf{L}\hat{\mathbf{w}}_{EU} = \frac{\mathbf{L}\hat{\mathbf{S}}_n^{-1}\mathbf{1}_p}{\mathbf{1}_p'\hat{\mathbf{S}}_n^{-1}\mathbf{1}_p} + \gamma^{-1}\mathbf{L}\hat{\mathbf{Q}}_n\bar{\mathbf{y}}_n,$$

$$\hat{\boldsymbol{\Omega}}_{\mathbf{L}} = \frac{\mathbf{L}\hat{\mathbf{Q}}_n\mathbf{L}'}{\mathbf{1}_p'\hat{\mathbf{S}}_n^{-1}\mathbf{1}_p} + \gamma^{-2}(\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n\bar{\mathbf{y}}_n + 1)\mathbf{L}\hat{\mathbf{Q}}_n\mathbf{L}' + \gamma^{-2}\mathbf{L}\hat{\mathbf{Q}}_n\bar{\mathbf{y}}_n\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n\mathbf{L}'$$

Under the null hypothesis:

- $T_{\mathbf{L}} \xrightarrow{d} \chi^2_k$ as $n \to \infty$ (classical asymptotics)

High-dimensional improvement

Test statistics:

$$T_{\mathbf{L};c} = (n-p)\left(\hat{\mathbf{w}}_{\mathbf{L};c} - \mathbf{r}\right)' \hat{\boldsymbol{\Omega}}_{\mathbf{L};c}^{-1} \left(\hat{\mathbf{w}}_{\mathbf{L};c} - \mathbf{r}\right),$$

where

$$\hat{\mathbf{w}}_{\mathbf{L};c} = \frac{\mathbf{L}\hat{\mathbf{S}}_n^{-1}\mathbf{1}_p}{\mathbf{1}_p'\hat{\mathbf{S}}_n^{-1}\mathbf{1}_p} + \gamma^{-1}\hat{s}_c\hat{\boldsymbol{\eta}}_{\mathbf{L};c},$$

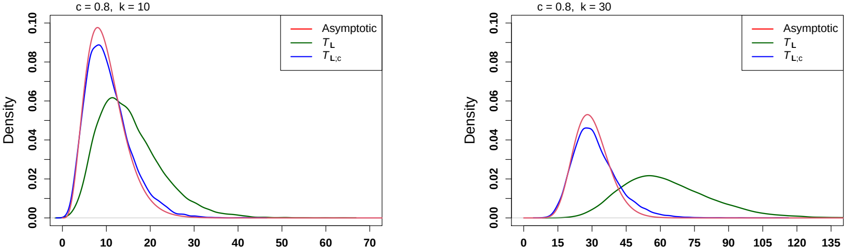
$$\hat{\boldsymbol{\Omega}}_{\mathbf{L};c} = \left(\gamma^{-2}(\hat{s}_c + 1) + \hat{V}_{GMV;c}\right)(1-c_n)\mathbf{L}\hat{\mathbf{Q}}\mathbf{L}^\top + \gamma^{-2}\hat{s}_c^2\hat{\boldsymbol{\eta}}_c\hat{\boldsymbol{\eta}}_c'$$

with

$$\hat{\boldsymbol{\eta}}_{\mathbf{L};c} = \frac{\hat{s}_c + c_n}{\hat{s}_c}\frac{\mathbf{L}\hat{\mathbf{Q}}_n\bar{\mathbf{y}}_n}{\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n\bar{\mathbf{y}}_n}, \quad \hat{s}_c = (1-c_n)\hat{s} - c_n, \quad \hat{V}_c = \frac{\hat{V}_{GMV}}{1-c_n}, \quad c_n = \frac{p}{n}$$

Under the null hypothesis:

- $T_{\mathbf{L};c} \xrightarrow{d} \chi_k^2$ for $p/n \to c$ as $n \to \infty$ with $k << p$ (high-dimensional asymptotics)



χ^2-approximation of the densities of $T_{\mathbf{L}}$ and $T_{\mathbf{L};c}$ together with their kernel density estimators for $\gamma = 5$ and $p = 300$

$c = 0.3$

	$k = 10$	$k = 30$	$k = 100$
$T_{\mathbf{L}}$	0.526	0.890	1
$T_{\mathbf{L};c}$	0.059	0.071	0.172

$c = 0.8$

	$k = 10$	$k = 30$	$k = 100$
$T_{\mathbf{L}}$	0.216	0.762	1
$T_{\mathbf{L};c}$	0.069	0.105	0.220

Empirical sizes of the tests based on $T_{\mathbf{L}}$ and $T_{\mathbf{L};c}$ using 10^4 independent replications.

Take home message

- Both $T_{\mathbf{L}}$ and $T_{\mathbf{L};c}$ are highly inconsistent
- Especially, $T_{\mathbf{L}}$ has a much higher size than the nominal significance value
- $T_{\mathbf{L};c}$ performs much better than $T_{\mathbf{L}}$ for smaller values of k, but discrepancy becomes large if k increases
- Both tests cannot be applied to validate the structure of the whole portfolio as a single test $\rightsquigarrow$ multiple test
- Negative impact on the power of each marginal test

Solution: New approaches for testing the structure of the whole optimal portfolio.

High-dimensional shrinkage test on $\mathbf{w}_{EU} = \mathbf{w}_0$

Hypotheses in terms of weights:

$$H_0 : \mathbf{w}_{EU} = \mathbf{w}_0 \qquad \text{against} \qquad H_1 : \mathbf{w}_{EU} \neq \mathbf{w}_0,$$

Idea: set $\mathbf{b} = \mathbf{w}_0 \rightsquigarrow \alpha^*(\mathbf{w}_0) = 0$

Theorem

Under the null hypothesis, it holds that

$$\sqrt{n}\hat{\alpha}_c^*(\mathbf{w}_0) \xrightarrow{d} \mathcal{N}(0, C_{\alpha;0}), \text{ for } p/n \to c \in [0,1) \text{ as } n \to \infty$$

- $\hat{\alpha}_c^*(\mathbf{w}_0)$ is a consistent estimator of $\alpha^*(\mathbf{w}_0)$
- $C_{\alpha;0} = C_{\alpha;0}(R_{GMV}, R_b, V_{GMV}, V_b, s)$.

Bodnar, T., Dmytriv, S., Okhrin, Y., Parolya, N., and Schmid, W. (2021). Statistical inference for the expected utility portfolio in high dimensions, *IEEE Transactions on Signal Processing* **69**: 1-14.

Test statistics

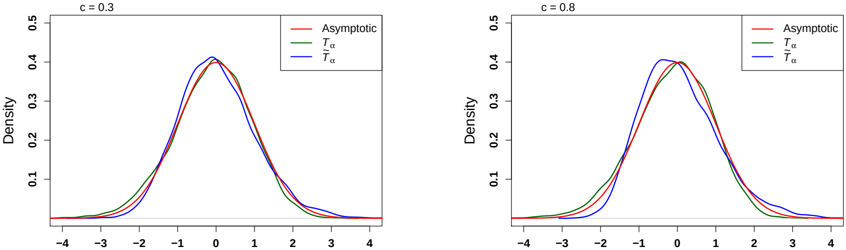
Under the null hypothesis:

- $$T_\alpha = \frac{\sqrt{n}\hat{\alpha}_c^*(\mathbf{w}_0)}{\sqrt{C_{\alpha;0}(\hat{R}_{GMV}, \hat{R}_b, \hat{V}_c, \hat{V}_b, \hat{s}_c)}} \xrightarrow{d} \mathcal{N}(0,1) \text{ for } p/n \to c \in [0,1) \text{ as } n \to \infty$$
- $$\widetilde{T}_\alpha = \frac{\sqrt{n}\hat{\alpha}_c^*(\mathbf{w}_0)}{\sqrt{C_{\alpha;0}(\hat{R}_{GMV}, \hat{R}_b, \hat{V}_c, \hat{V}_b, \gamma(\hat{R}_b - \hat{R}_{GMV}))}} \xrightarrow{d} \mathcal{N}(0,1) \text{ for } p/n \to c \in [0,1) \text{ as } n \to \infty$$

  where
  - $\hat{R}_{GMV}$, $\hat{R}_b$, and $\hat{V}_b$ are the sample estimator of R_{GMV}, R_b, and V_b
  - $\hat{V}_c$ and $\hat{s}_c$ are consistent estimators of V_{GMV} and s

Remark: Using the duality between the test theory and confidence interval, the null hypothesis is rejected at significance level β as soon as the $(1-\beta)$ confidence interval constructed for $\alpha^*(\mathbf{w}_0)$ does not include zero.

Size properties



Normal approximation of the densities of T_α and $\widetilde{T}_\alpha$ together with their kernel density estimators for $\gamma = 5$ and $p = 300$

	$\mathbf{c = 0.3}$	$\mathbf{c = 0.8}$
T_α	0.052	0.053
$\tilde{T}_\alpha$	0.051	0.052

Empirical sizes of the tests based on T_α and $\tilde{T}_\alpha$ using 10^4 independent replications.

Power analysis

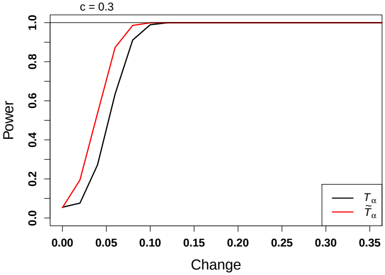
Model under the alternative hypothesis

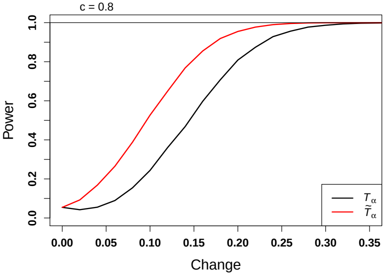
$$\boldsymbol{\mu}_1 = \boldsymbol{\mu} + \epsilon,$$

where

$$\epsilon = -a \cdot (\underbrace{1, \ldots, 1}_{m}, \underbrace{0, \ldots, 0}_{m}),$$

where $a = 0.01\kappa,\ \kappa \in \{0, 1, 2, \ldots, 35\}$, $m = 0.5p$.



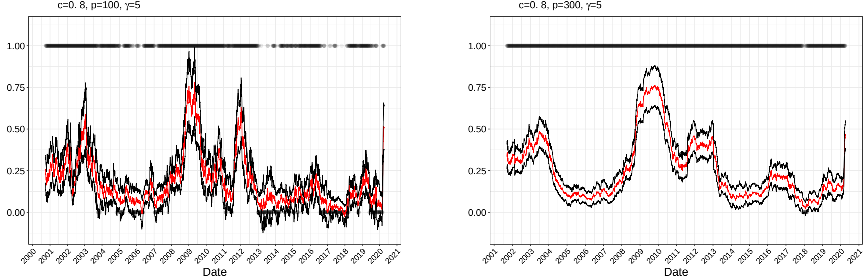


Empirical power of the proposed tests as a function of the change a and $p = 300$

Efficiency of the equally weighted portfolio: empirical illustration

Data: Daily returns on all companies listed in the S&P 500 index for the period from April 1999 to March 2020

Portfolio: First p assets, $p \in \{100, 300\}$, in alphabetic order from the available data.

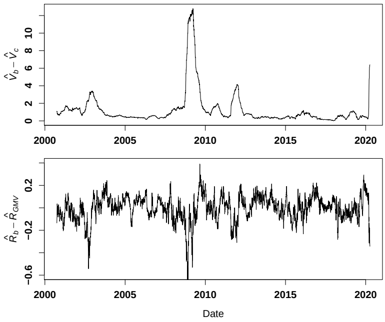


Estimated shrinkage intensities for the equally weighted portfolio as the target portfolio with 95% pointwise confidence intervals. The black dots indicate the periods with rejected H_0 (1-values) and not rejected H_0 (0-values)

Bodnar, T., S. Dmytriv, Y. Okhrin, D. Otryakhin and N. Parolya (2022) HDShOP: High-Dimensional Shrinkage Optimal Portfolios. *R package version 0.1.3*.

Why all these efforts?

- Interesting mathematics
- Better understanding the properties of the estimators and the test statistics



- Simple application to financial data
- Interesting empirical results
  - Equally weighted portfolio is not efficient, especially when p is large
  - Naïve portfolio has a poor performance during the volatile period

Thank you very much for your attention!