ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

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B-1201 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

If $a, b, c > 0$, then prove that, for any positive integer n,

$$\frac{a^3}{aF_n + bF_{n+1}} + \frac{b^3}{bF_n + aF_{n+1}} \ge \frac{a^2+b^2}{F_{n+2}},$$
$$\frac{a^3}{aL_n + bL_{n+1}} + \frac{b^3}{bL_n + aL_{n+1}} \ge \frac{a^2+b^2}{L_{n+2}},$$
$$\frac{a^3}{aF_n + bF_{n+1} + cF_{n+2}} + \frac{b^3}{bF_n + cF_{n+1} + aF_{n+2}} + \frac{c^3}{cF_n + aF_{n+1} + bF_{n+2}} \ge \frac{a^2+b^2+c^2}{2F_{n+2}},$$
$$\frac{a^3}{aL_n + bL_{n+1} + cL_{n+2}} + \frac{b^3}{bL_n + cL_{n+1} + aL_{n+2}} + \frac{c^3}{cL_n + aL_{n+1} + bL_{n+2}} \ge \frac{a^2+b^2+c^2}{2L_{n+2}}.$$

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MAY 2017

B-1208 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

For every positive integer n, find all real solutions of the following linear system of equations:

$$\begin{array}{ccccccccccccc}
F_1x_1 & + & x_2 & & & & & & & & & = & F_3, \\
F_2x_1 & + & F_1x_2 & + & x_3 & & & & & & & = & F_4, \\
F_3x_1 & + & F_2x_2 & + & F_1x_3 & + & \cdots & & & & & = & F_5, \\
\vdots & & \vdots & & \vdots & & & \ddots & & & & & \vdots \\
F_{n-1}x_1 & + & F_{n-2}x_2 & + & F_{n-3}x_3 & + & \cdots & + & x_n & & & = & F_{n+1}, \\
F_nx_1 & + & F_{n-1}x_2 & + & F_{n-2}x_3 & + & \cdots & + & F_1x_n & + & x_{n+1} & = & F_{n+2}, \\
F_{n+1}x_1 & + & F_nx_2 & + & F_{n-1}x_3 & + & \cdots & + & F_2x_n & + & F_1x_{n+1} & = & F_{n+3} - 1.
\end{array}$$

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AUGUST 2017

B-1213 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

For every positive integer n, prove that

$$\frac{F_1}{F_3}\cdot\frac{F_5}{F_7}\cdot\ \cdots\ \cdot\frac{F_{4n-3}}{F_{4n-1}} > \sqrt[4]{\frac{1}{F_1+F_5+\cdots+F_{8n+1}}},$$

and

$$\frac{F_2}{F_4}\cdot\frac{F_6}{F_8}\cdot\ \cdots\ \cdot\frac{F_{4n-2}}{F_{4n}} < \sqrt[4]{\frac{2}{F_3+F_7+\cdots+F_{8n+3}}}.$$

SOLUTIONS

Polygon with Generalized Fibonacci Numbers as Its Vertices

B-1195 Proposed by Jeremiah Bartz, Francis Marion University, Florence, SC. (Vol. 54.3, August 2016)

Let G_i denote the generalized Fibonacci sequence given by $G_0 = a$, $G_1 = b$, and $G_i = G_{i-1} + G_{i-2}$ for $i \geq 3$. Let $m \geq 0$ and $k \geq 0$. Prove that the area A of the polygon with $n \geq 3$ vertices

$$(G_m, G_{m+k}), (G_{m+2k}, G_{m+3k}), \ldots, (G_{m+(2n-2)k}, G_{m+(2n-1)k})$$

is

$$\frac{|\mu| F_k\big(F_{2k(n-1)} - (n-1)F_{2k}\big)}{2}$$

where $\mu = a^2 + ab - b^2$.

Solution by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

It is known [1, Theorem 33.3] that the area with vertices (G_n, G_{n+r}), (G_{n+p}, G_{n+p+r}), and (G_{n+q}, G_{n+q+r}) is independent of n, and equals

$$\frac{1}{2}\left|\mu F_r\big((-1)^p F_{q-p} + F_p - F_q\big)\right|.$$

For $n = m$, $r = k$, $p = 2ks$, where $1 \leq s \leq n-2$, and $q = p + 2k$, we obtain the area

$$\frac{1}{2}\left|\mu F_k\big(F_{2k} + F_{2ks} - F_{2k(s+1)}\big)\right| = \frac{1}{2}|\mu| F_k\big(F_{2k(s+1)} - F_{2ks} - F_{2k}\big).$$

Therefore,

$$\begin{aligned} A &= \frac{1}{2}|\mu| F_k \sum_{s=1}^{n-2}\big(F_{2k(s+1)} - F_{2ks} - F_{2k}\big) \\ &= \frac{1}{2}|\mu| F_k\big(F_{2k(n-1)} - F_{2k} - (n-2)F_{2k}\big) \\ &= \frac{1}{2}|\mu| F_k\big(F_{2k(n-1)} - (n-1)F_{2k}\big). \end{aligned}$$

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley, New York, 2001.

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B-1218 **Proposed by Ivan V. Fedak, Vasyl Stefanyk Precapathian National University, Ivano-Frankivsk, Ukraine.**

Find a closed form expression for

$(L_{n+1}-1)F_n(F_{2n+2}-F_{n+2})+(1-F_n-F_{n+2})F_{n+2}(F_{2n+2}-F_{n+3})+(F_{2n+2}-F_{n+2})(F_{2n+2}-F_{n+3})$.

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B-1223 **Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.**

For all positive integers n and a, prove that

$$\sum_{k=1}^{n} F_k(F_{k+1}^a + F_{k+2}^a - F_{n+2}^a - 1) \leq 0.$$

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SOLUTIONS

Two Doses of AM-GM Inequality

B-1206 **Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain. (Vol. 55.2, May 2017)**

Let $n \geq 2$ be an integer. Prove that

$$1 + \frac{1}{n^2} \sum_{1\leq i<j\leq n} \frac{\left(\sqrt{F_iF_{j+1}} - \sqrt{F_{i+1}F_j}\right)^2}{F_iF_j} \leq \frac{1}{n}\sum_{k=1}^{n} \frac{F_{k+1}}{F_k},$$

in which the subscripts are taken modulo n.

Solution by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

To begin, we rewrite the given inequality in the form

$$n^2 + \sum_{1\leq i<j\leq n} \left(\frac{F_{i+1}}{F_i} + \frac{F_{j+1}}{F_j} - 2\sqrt{\frac{F_{i+1}}{F_j} \cdot \frac{F_{j+1}}{F_j}} \right) \leq n \cdot \sum_{k=1}^{n} \frac{F_{k+1}}{F_k}.$$

Next, we see that

$$\sum_{1\leq i<j\leq n} \left(\frac{F_{i+1}}{F_i} + \frac{F_{j+1}}{F_j} \right) = (n-1)\sum_{k=1}^{n} \frac{F_{k+1}}{F_k},$$

$$\sum_{1\leq i<j\leq n} \sqrt{\frac{F_{i+1}}{F_i} \cdot \frac{F_{j+1}}{F_j}} \geq \frac{n(n-1)}{2} \cdot \sqrt[\frac{n(n-1)}{2}]{\prod_{1\leq i<j\leq n} \sqrt{\frac{F_{i+1}}{F_i} \cdot \frac{F_{j+1}}{F_j}}} = \frac{n(n-1)}{2},$$

$$\sum_{k=1}^{n} \frac{F_{k+1}}{F_k} \geq n \cdot \sqrt[n]{\prod_{k=1}^{n} \frac{F_{k+1}}{F_k}} = n,$$

in which the subscripts are taken modulo n. From it, follows that

$$\begin{aligned} n^2 + \sum_{1 \leq i < j \leq n} \left(\frac{F_{i+1}}{F_j} + \frac{F_{j+1}}{F_i} - 2\sqrt{\frac{F_{i+1}}{F_i} \cdot \frac{F_{j+1}}{F_j}} \right) &\leq n^2 + (n-1)\sum_{k=1}^{n} \frac{F_{k+1}}{F_k} - 2 \cdot \frac{n(n-1)}{2} \\ &= n + (n-1)\sum_{k=1}^{n} \frac{F_{k+1}}{F_k} \\ &\leq \sum_{k=1}^{n} \frac{F_{k+1}}{F_k} + (n-1)\sum_{k=1}^{n} \frac{F_{k+1}}{F_k} \\ &= n \cdot \sum_{k=1}^{n} \frac{F_{k+1}}{F_k}. \end{aligned}$$

Row Reduction on an Augmented Matrix

B-1208 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.
(Vol. 55.2, May 2017)

For every positive integer n, find all real solutions of the following linear system of equations:

$$\begin{array}{rcrcrcccrcrcl} F_1x_1 & + & x_2 & & & & & & & & & = & F_3, \\ F_2x_1 & + & F_1x_2 & + & x_3 & & & & & & & = & F_4, \\ F_3x_1 & + & F_2x_2 & + & F_1x_3 & + & \cdots & & & & & = & F_5, \\ \vdots & & \vdots & & \vdots & & & \ddots & & & & & \vdots \\ F_{n-1}x_1 & + & F_{n-2}x_2 & + & F_{n-3}x_3 & + & \cdots & + & x_n & & & = & F_{n+1}, \\ F_nx_1 & + & F_{n-1}x_2 & + & F_{n-2}x_3 & + & \cdots & + & F_1x_n & + & x_{n+1} & = & F_{n+2}, \\ F_{n+1}x_1 & + & F_nx_2 & + & F_{n-1}x_3 & + & \cdots & + & F_2x_n & + & F_1x_{n+1} & = & F_{n+3} - 1. \end{array}$$

Composite solution by the proposer and the Elementary Problems Editor.

We can use Gauss-Jordan elimination to solve the linear system. We want to apply row reduction to the following augmented matrix:

$$\left[\begin{array}{cccccc|c} F_1 & 1 & 0 & \cdots & 0 & 0 & F_3 \\ F_2 & F_1 & 1 & \cdots & 0 & 0 & F_4 \\ F_3 & F_2 & F_1 & \cdots & 0 & 0 & F_5 \\ \vdots & \vdots & \vdots & \cdots & & & \vdots \\ F_{n-1} & F_{n-2} & F_{n-3} & \cdots & 1 & 0 & F_{n+1} \\ F_n & F_{n-1} & F_{n-2} & \cdots & F_1 & 1 & F_{n+2} \\ F_{n+1} & F_n & F_{n-1} & \cdots & F_2 & F_1 & F_{n+3} - 1 \end{array}\right].$$

For $k = 3, 4, \ldots, n+1$, subtracting the sum of the first $k-2$ rows from row k reduces the augmented matrix to (recall that $F_1 + F_2 + \cdots + F_m = F_{m+2} - 1$)

$$\left[\begin{array}{ccccccccc|c} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 3 \\ 1 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 3 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 2 \end{array}\right].$$

For $k = n, n-1, \ldots, 2, 1$, subtracting row k from row $k+1$ further reduces the augmented matrix to

$$\left[\begin{array}{ccccccccc|c} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & -1 \end{array}\right].$$

We deduce that $x_2 = 2 - x_1$, $x_3 = x_n = 1$, and $x_k = x_{k+2}$ for $k = 2, 3, \ldots, n-1$. Therefore, if n is even, the solution is

$$x_1 = x_2 = \cdots = x_{n+1} = 1;$$

but if n is odd, the solution is

$$x_1 = t, \quad x_2 = x_4 = \cdots = x_{n+1} = 2 - t, \quad x_3 = x_5 = \cdots = x_n = 1,$$

where t is an arbitrary real number.

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B-1234 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

Let $n \geq 3$ be an odd integer. Find the real solutions of the following system of equations:

$$\begin{aligned} x_1^3 + x_1 + x_2 &= F_1 x_1^2 + F_3, \\ x_2^5 + x_2 + x_3 &= F_2 x_2^4 + F_4, \\ &\vdots \\ x_{n-1}^{2n-1} + x_{n-1} + x_n &= F_{n-1} x_{n-1}^{2n-2} + F_{n+1}, \\ x_n^{2n+1} + \frac{F_{n+2} - 1}{F_n} x_n + x_1 &= F_n x_n^{2n} + F_{n+2}. \end{aligned}$$

SOLUTIONS

An Intriguing Telescoping Product

B-1213 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.
(Vol. 55.3, August 2017)

For every positive integer n, prove that

$$\frac{F_1}{F_3} \cdot \frac{F_5}{F_7} \cdot \cdots \cdot \frac{F_{4n-3}}{F_{4n-1}} > \sqrt[4]{\frac{1}{F_1 + F_5 + \cdots + F_{8n+1}}},$$

and

$$\frac{F_2}{F_4} \cdot \frac{F_6}{F_8} \cdot \cdots \cdot \frac{F_{4n-2}}{F_{4n}} < \sqrt[4]{\frac{2}{F_3 + F_7 + \cdots + F_{8n+3}}}.$$

Solution by the proposer.

If $k < m$, then for every positive integer p, we find, by means of Binet's formula, or by applying Identity 2 in [1, page 87],

$$F_k F_{m+p} - F_{k+p} F_m = (-1)^{k+1} F_p F_{m-k}.$$

Hence, $\frac{F_k}{F_m} > \frac{F_k+p}{F_{m+p}}$ if k is odd, and $\frac{F_k}{F_m} < \frac{F_k+p}{F_{m+p}}$ if k is even. Thus,

$$\frac{F_1}{F_3} \cdot \frac{F_5}{F_7} \cdot \cdots \cdot \frac{F_{4n-3}}{F_{4n-1}} > \frac{F_2}{F_4} \cdot \frac{F_6}{F_8} \cdot \cdots \cdot \frac{F_{4n-2}}{F_{4n}},$$

$$\frac{F_1}{F_3} \cdot \frac{F_5}{F_7} \cdot \cdots \cdot \frac{F_{4n-3}}{F_{4n-1}} > \frac{F_3}{F_5} \cdot \frac{F_7}{F_9} \cdot \cdots \cdot \frac{F_{4n-1}}{F_{4n+1}},$$

$$\frac{F_1}{F_3} \cdot \frac{F_5}{F_7} \cdot \cdots \cdot \frac{F_{4n-3}}{F_{4n-1}} > \frac{F_4}{F_6} \cdot \frac{F_8}{F_{10}} \cdot \cdots \cdot \frac{F_{4n}}{F_{4n+2}}.$$

Therefore,

$$\left(\frac{F_1}{F_3} \cdot \frac{F_5}{F_7} \cdot \cdots \cdot \frac{F_{4n-3}}{F_{4n-1}}\right)^4 > \frac{F_1 F_2}{F_{4n+1}F_{4n+2}} = \frac{1}{\sum_{k=1}^{4n+1} F_k^2}.$$

The first inequality follows from

$$\sum_{k=1}^{4n+1} F_k^2 = F_1^2 + (F_2^2 + F_3^2) + \cdots + (F_{4n}^2 + F_{4n+1}^2) = F_1 + F_5 + \cdots + F_{8n+1}.$$

The second inequality can be obtained in a similar manner.

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, New York, 2001.

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B-1239 **Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.**

For all integers n, prove that

$$\left(\frac{1}{L_n} - \frac{1}{L_{n+1}}\right)^4 + \left(\frac{1}{L_{n+1}} + \frac{1}{L_{n+2}}\right)^4 + \left(\frac{1}{L_{n+2}} + \frac{1}{L_n}\right)^4 = 2\left(\frac{1}{L_n} + \frac{1}{L_{n+1}} - \frac{1}{L_{n+2}}\right)^4.$$

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B-1241 **Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.**

For all positive integers n, prove that

$$\frac{F_{n+2}}{L_{n+2}} + \frac{F_{n+1}}{L_{n+1}} + \frac{F_n}{L_{n+1} + F_{n+2}} > 1.$$

Determinant of a Symmetric Matrix

B-1221 **Proposed by José Luis Díaz-Barrero, Technical University of Catalonia (Barcelona Tech), Barcelona Spain.**
(Vol. 56.1, February 2018)

For any positive integer n, show that

$$\frac{1}{54F_{2n}} \begin{vmatrix} 4 & F_n & L_n \\ F_n & (F_{n+1} + L_n)^2 & F_{2n} \\ L_n & F_{2n} & F_{n+2}^2 \end{vmatrix}$$

is a perfect square, and find its value.

Composite solution by I. V. Fedak, Vasyl Stefanyk Precarpathian National University. Ivano-Frankivsk, Ukraine, and the editor.

Let $x = F_{n-1}$ and $y = F_{n+1}$, so that

$$F_n = y - x, \quad L_n = y + x, \quad F_{n+2} = 2y - x, \quad F_{n+1} + L_n = 2y + x,$$

and

$$F_{2n} = F_n L_n = (y-x)(y+x) = y^2 - x^2.$$

Then,

$$\begin{vmatrix} 4 & F_n & L_n \\ F_n & (F_{n+1}+L_n)^2 & F_{2n} \\ L_n & F_{2n} & F_{n+2}^2 \end{vmatrix} = \begin{vmatrix} 4 & y-x & y+x \\ y-x & (2y+x)^2 & y^2-x^2 \\ y+x & y^2-x^2 & (2y-x)^2 \end{vmatrix}$$
$$= 4(2y+x)^2(2y-x)^2 - 2(y^2-x^2)^2 - (y+x)^2(2y+x)^2 - (y-x)^2(2y-x)^2,$$

which can be simplified to

$$\begin{aligned} &\left[(2(2y-x)^2 - (y+x)^2\right](2y+x)^2 + \left[2(2y+x)^2 - (y-x)^2\right](2y-x)^2 - 2(y^2-x^2)^2 \\ &= (7y^2 - 10xy + x^2)(2y+x)^2 + (7y^2 + 10xy + x^2)(2y-x)^2 - 2(y^2-x^2)^2 \\ &= 2(7y^2 + x^2)(4y^2 + x^2) - 80x^2y^2 - 2(y^2-x^2)^2 \\ &= 54y^2(y^2 - x^2) \\ &= 54F_{n+1}^2 F_{2n}. \end{aligned}$$

Thus, the value of the given expression is F_{n+1}^2 for any positive inteber n.

An Inequality with a Geometric Twist

B-1223 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.
(Vol. 56.1, February 2018)

For all positive integers n and a, prove that

$$\sum_{k=1}^{n} F_k(F_{k+1}^a + F_{k+2}^a - F_{n+2}^a - 1) \le 0.$$

Solution 1 by Wei-Kai Lai, University of South Carolina Salkehatchie, Walterboro, SC.

The claimed inequality is equivalent to

$$\sum_{k=1}^{n} F_k(F_{k+1}^a + F_{k+2}^a) \le (F_{n+2}^a + 1)\sum_{k=1}^{n} F_k = (F_{n+2}^a + 1)(F_{n+2} - 1).$$

We find

$$\sum_{k=1}^{n} F_k(F_{k+1}^a + F_{k+2}^a) = F_1F_2^a + F_nF_{n+2}^a + \sum_{k=1}^{n-1}(F_k + F_{k+1})F_{k+2}^a = 1 + F_nF_{n+2}^a + \sum_{k=1}^{n-1} F_{k+2}^{a+1}.$$

Since $F_1 = F_2 = 1$, we can further rewrite the claimed inequality as

$$F_nF_{n+2}^a + \sum_{i=1}^{n+1} F_i^{a+1} \le F_{n+2}^{a+1} - F_{n+2}^a + F_{n+2},$$

or

$$\sum_{i=1}^{n+1} F_i^{a+1} \le F_{n+2}^a F_{n+1} - F_{n+2}^a + F_{n+2}.$$

We will prove this inequality by induction on n. The equality becomes an equality when $n = 1$. Assume it is true when $n = k$. Then,

$$\sum_{i=1}^{k+2} F_i^{a+1} \le F_{k+2}^a F_{k+1} - F_{k+2}^a + F_{k+2} + F_{k+2}^{a+1}.$$

To complete the inductive step, it suffices to prove that

$$F_{k+2}^aF_{k+1} - F_{k+2}^a + F_{k+2} + F_{k+2}^{a+1} \le F_{k+3}^aF_{k+2} - F_{k+3}^a + F_{k+3},$$

or equivalently,

$$F_{k+1}(F_{k+2}^a - 1) \le (F_{k+2} - 1)(F_{k+3}^a - F_{k+2}^a).$$

After factoring $F_{k+2}^a - 1$ and $F_{k+3}^a - F_{k+2}^a$ and canceling common factors, the inequality above reduces to

$$\sum_{j=0}^{a-1} F_{k+2}^{a-1-j} \le \sum_{j=0}^{a-1} F_{k+3}^{a-1-j}F_{k+2}^j,$$

which is obviously true. Therefore, the claimed inequality is true for any positive integer n.

Solution 2 by the proposer.

The inequality becomes an equality when $n = 1$, so we shall assume $n > 1$. Using $F_2 = 1$, and the identities $F_k = F_{k+2} - F_{k+1}$ and $\sum_{k=1}^n F_k = F_{n+2} - 1$, we can write the given inequality as

$$\sum_{k=1}^n (F_{k+2} - F_{k+1}) \cdot \frac{F_{k+1}^a + F_{k+2}^a}{2} \le (F_{n+2} - F_2) \cdot \frac{F_{n+2}^a + F_2^a}{2}.$$

Let A_k denote the point $\left(F_k, F_k^a\right)$ on the graph of the function $f(x) = x^a$, and B_k denote the point $(F_k, 0)$. The left side is the sum of the areas of the trapezoids $A_{k+1}A_{k+2}B_{k+2}B_{k+1}$ from $k = 1$ to $k = n$. The right side of the inequality above is the area of the trapezoid $A_2A_{n+2}B_{n+2}B_2$. Because $f(x) = a^x$ is a convex function, it is obvious that the left side is less than or equal to the right side.

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Another Hassenberg Matrix Problem

B-1230 **Proposed by T. Goy, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.**

For all integers $n \ge 0$, prove that

$$F_{2n+1} = (-1)^n \sum_{\substack{t_1,t_2,\ldots,t_n \ge 0 \\ t_1+2t_2+\cdots+nt_n=n}} (-1)^{t_1+t_3+\cdots+t_{n-[1+(-1)^n]/2}} \frac{(t_1+t_2+\cdots+t_n)!}{t_1!\,t_2!\,\cdots\,t_n!} \cdot 2^{t_1}.$$

Solution by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

We consider the Hassenberg matrix

$$H_n = \begin{pmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{pmatrix}, \qquad a_0 \ne 0.$$

It is known that

$$\det(H_n) = \sum_{\substack{t_1,t_2,\ldots,t_n \ge 0 \\ t_1+2t_2+\cdots+nt_n=n}} (-a_0)^{n-(t_1+t_2+\cdots+t_n)} \frac{(t_1+t_2+\cdots+t_n)!}{t_1!\,t_2!\,\cdots\,t_n!} a_1^{t_1}a_2^{t_2}\cdots a_n^{t_n}.$$

Let $a_0 = 1$, $a_1 = 2$,

$$a_2 = a_4 = \cdots = a_{2k} = -1, \quad 2k \le n, \qquad \text{and} \qquad a_3 = a_5 = \cdots = a_{2k+1} = 1, \quad 2k+1 \le n.$$

Then, we obtain

$$\begin{aligned} S_n &= (-1)^n \sum_{\substack{t_1,t_2,\ldots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} (-1)^{t_1+t_3+\cdots+t_{n-[1+(-1)^n]/2}} \frac{(t_1+t_2+\cdots+t_n)!}{t_1!\,t_2!\,\cdots\,t_n!}\cdot 2^{t_1} \\ &= \det\begin{pmatrix} 2 & 1 & 0 & \cdots & 0 & 0\\ -1 & 2 & 1 & \cdots & 0 & 0\\ 1 & -1 & 2 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ (-1)^{n-2} & (-1)^{n-3} & (-1)^{n-4} & \cdots & 2 & 1\\ (-1)^{n-1} & (-1)^{n-2} & (-1)^{n-3} & \cdots & -1 & 2 \end{pmatrix}. \end{aligned}$$

Adding row $k-1$ to row k in this matrix for $k=n$ to $k=2$, we get that this determinant equals

$$\det\begin{pmatrix} 2 & 1 & 0 & \cdots & 0 & 0\\ 1 & 3 & 1 & \cdots & 0 & 0\\ 0 & 1 & 3 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 3 & 1\\ 0 & 0 & 0 & \cdots & 1 & 3 \end{pmatrix} = 2\det(A_{n-1}) - \det(A_{n-2}),$$

where

$$A_n = \begin{pmatrix} 3 & 1 & 0 & \cdots & 0 & 0\\ 1 & 3 & 1 & \cdots & 0 & 0\\ 0 & 1 & 3 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 3 & 1\\ 0 & 0 & 0 & \cdots & 1 & 3 \end{pmatrix}_{n\times n}.$$

It is an easy exercise to find the recurrence relation $\det(A_n) = 3\det(A_{n-1}) - \det(A_{n-2})$, from which we determine that $\det(A_n) = F_{2n+2}$. Therefore,

$$S_n = 2F_{2n} - F_{2n-2} = F_{2n+1}.$$

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Two Fibonacci-Lucas Identities with Central Binomial Coefficient

B-1231 **Proposed by Kenny B. Davenport, Dallas, PA.**
(Vol. 56.3, August 2018)

Find the closed form expressions for the sums

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{nF_n}{8^n}, \qquad \text{and} \qquad \sum_{n=1}^{\infty}\binom{2n}{n}\frac{nL_n}{8^n}.$$

Solution by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

The generating function for the Catalan numbers is known to be

$$g(x) = \sum_{n=1}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n = \frac{1-\sqrt{1-4x}}{2x}.$$

From here it follows that

$$h(x) = \sum_{n=0}^{\infty}\binom{2n}{n}x^n = \frac{d}{dx}\left[x\,g(x)\right] = \frac{d}{dx}\left(\frac{1-\sqrt{1-4x}}{2}\right) = \frac{1}{\sqrt{1-4x}}.$$

Therefore,

$$f(x) = \sum_{n=0}^{\infty}\binom{2n}{n}nx^n = x\,h'(x) = \frac{2x}{\sqrt{(1-4x)^3}}.$$

Thus, using Binet's formulas, we find

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{nF_n}{8^n}=\frac{1}{\sqrt{5}}\left[f\left(\frac{\alpha}{8}\right)-f\left(\frac{\beta}{8}\right)\right]=\frac{1}{\sqrt{10}}\left[\frac{\alpha}{\sqrt{(2-\alpha)^3}}-\frac{\beta}{\sqrt{(2-\beta)^3}}\right].$$

Finally, use the identities $2-\alpha=\beta^2$ and $2-\beta=\alpha^2$ to obtain (recall that $\beta<0$)

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{nF_n}{8^n}=\frac{1}{\sqrt{10}}\left(\frac{\alpha}{-\beta^3}-\frac{\beta}{\alpha^3}\right)=\frac{\alpha^4+\beta^4}{\sqrt{10}}=\frac{L_4}{\sqrt{10}}=\frac{7}{\sqrt{10}}.$$

Similarly,

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{nL_n}{8^n}=f\left(\frac{\alpha}{8}\right)+f\left(\frac{\beta}{8}\right)=\frac{1}{\sqrt{2}}\left(\frac{\alpha}{-\beta^3}+\frac{\beta}{\alpha^3}\right)=\frac{\alpha^4-\beta^4}{\sqrt{2}}=\frac{5F_4}{\sqrt{10}}=\frac{15}{\sqrt{10}}.$$

A Convoluted System of Equations

B-1234 **Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.**
(Vol. 56.3, August 2018)

Let $n\geq 3$ be an odd integer. Find the real solutions of the following system of equations:

$$\begin{aligned}
x_1^3+x_1+x_2 &= F_1x_1^2+F_3,\\
x_2^5+x_2+x_3 &= F_2x_2^4+F_4,\\
&\vdots\\
x_{n-1}^{2n-1}+x_{n-1}+x_n &= F_{n-1}x_{n-1}^{2n-2}+F_{n+1},\\
x_n^{2n+1}+\frac{F_{n+2}-1}{F_n}x_n+x_1 &= F_nx_n^{2n}+F_{n+2}.
\end{aligned}$$

Solution by the Proposer.

We use $F_{k+2}=F_{k+1}+F_k$, and write the system of equation as

$$\begin{aligned}
(x_1-F_1)(x_1^2+1) &= F_2-x_2,\\
(x_2-F_2)(x_2^4+1) &= F_3-x_3,\\
&\vdots\\
(x_{n-1}-F_{n-1})(x_{n-1}^{2n-2}+1) &= F_{n-1}-x_{n-1},\\
(x_n-F_n)\left(x_n^{2n}+\frac{F_{n+2}-1}{F_n}\right) &= F_1-x_1.
\end{aligned}$$

If $x_1>F_1$, then $x_2<F_2$; thus, $x_3>F_3$, etc. Since n is odd, this ends with $x_n>F_n$, which in turn implies that $x_1<F_1$. This contradiction asserts that $x_1\leq F_1$. Likewise, we also have $x_1\geq F_1$. Therefore, $x_1=F_1$. Consequently, $x_k=F_k$ for $k=1,2,\ldots,n$.

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B-1256 **Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk,Ukraine.**

For any positive integers n find an infinite set of pairs of positive Fibonacci numbers x and y such that $x^2-xy-y^2=F_nF_{n+1}-F_{n-1}F_{n+2}$.